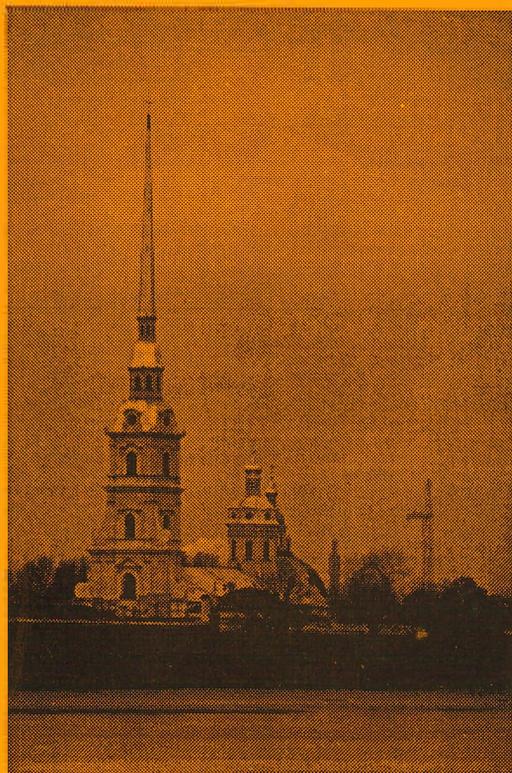# Report of the Exchange Programme of ILIP., St. Petersburg and Eulers Venner, Aarhus.

Part 3: The Stay in Russia seen with Danish Eyes. October 18-25 1992.

Eulers Venner
Matematisk Institut
Aarhus University.

Report of the Exchange Program of ILIP, St. Petersburg and Eulers Venner, Aarhus.

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Contents

Preface

Good human relationships is what counts.
What should we do without them?

Gennady Khorkov. St. Petersburg 1992.

At this place we first of all thank Joint Stock Enterprise ILIP for the smooth cooperation during the entire exchange programme, but in particular for the actual implementation of our stay in Russia. However, the success of our visit was the result of the efforts by many. In that light we especially would like to thank Yekaterina Mustafyna, Elena Roubintshik, Alexey Senichev, Konstantin Egorov and Andrey Solovyev for putting in so much time and effort to make the week such a great success.

Of course we also thank the generous and hospitable Russian families who where so kind to accommodate some of the Danes: Stanislav Krainikov, Andrey Solovyev, Alexey Rodionenko, Yaroslav Suchkov, Vladimir Parkman, Pavel Zharov, Natalia Tomashkova and Timophey Sumyn.

It would also be in its place to thank 'Udenrigsministeriets Demokratifond', without whom it would not have been possible for the Russians to travel to Denmark. Their visit here resulted in many close friendships between Russians and Danes, and without these friends we Danes would not have had quite as pleasant a stay in St. Petersburg as we did.

Naturally the realization of this project would have been utterly impossible without the generously donated financial aid by various sources;

Finally we thank everybody else who has participated in the exchange program in whatsoever manner and making it the pleasant experience it turned out to be.

On behalf of the Danish group

Pieter van de Griend   Lars P. G. Westergaard   Eva Kathrine Petersen

Sunday 18 October 1992

16.40 Meeting at the airport Pulkovo.

17.30 Accommodation in the hostel and with families.

18.00 Dinner.

This day our trip began. We got up very early to catch our train from Århus to Copenhagen. Not everybody was happy about the choice of departure time. Mostly because 5 a.m is too early to be happy at all, but also because the bad jokes about the Aeroflot planes were becoming annoying. However, as dawn broke, the mood became more relaxed, and we began to realize that the tour we had been waiting for so long had really begun.

At Copenhagen Airport

The journey turned out to be smooth until we landed at St. Petersburg's airport. First we waited for at least 20 minutes before we were let out of the plane. Then we waited for an hour and a half to get through customs. This was somewhat annoying, especially after we saw the Russian students waiting for us at the other side of the custom check, but after roughly 90 minutes of patience we were released. Our Russian friends welcomed us and took us to our hotel by bus. Here we got our rooms and we tried our first Russian meal.

Monday 19 October 1992

8.00 — 9.00 Breakfast.

9.50 — 10.10 Meeting with the Head of Mathematics department of the Electrical Engineering University.

10.10 — 11.40 Lecture on polynomial- and rational approximation of analytic functions on planar sets.

12.30 — 13.00 Meeting with the leading staff of ILIP.

13.00 — 14.00 Lunch at the Palace of Youth.

14.00 — 17.30 Sight-seeing round the city.

18.00 — Dinner at the Palace of Youth.

Welcome by the Head of the Mathematics Department of the Electrical Engineering University.

The head of the Mathematics Department welcomed us on behalf of St. Petersburg University of Electrical Engineering and gave us an impression of the institute's history and the buildings in which it resides. Currently the university has approximately 6000 students and 145 lecturers. He concluded his brief welcome by wishing us a pleasant and educational stay.

The head of the Mathematics Department

Mr. Dyakonov

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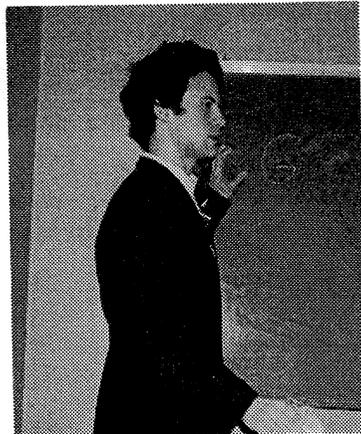
Lecture on Polynomial- and Rational Approximations of Analytic Functions by Konstantin Dyakonov

Let K be a compact subset of C. Let $A(K)$ denote the continuous complex functions on K which are holomorphic in K^0. If $K^0 = \emptyset$ then $A(K)$ simply denotes the continuous complex functions on K. $A(K) \subseteq C(K)$ is a vectorspace and we equip it with the norm $||f|| = \max\{|f(z)| : z \in K\}$

We consider the two questions:

1. Which K have the property that for any $f \in A(K)$ there exists a sequence (P_n) of polynomials such that $||f - P_n|| \to 0$ as $n \to \infty$ (polynomial approximation).

2. The same, but with (R_n), a sequence of rational functions (rational approximation). By a rational function we simply mean a function of the form $\frac{P(z)}{Q(z)}$ where P and Q are polynomials.

We will start by making a stronger assumption. We say that f is holomorphic on K iff there exists an open neighborhood G of K so that $f : G \to$ C is holomorphic. We have the following theorem:

Theorem 1 (Runge's first theorem 1885): *Any function which is holomorphic on K can be approximated by a rational function.*

The theorem is proved by using Cauchy's formula on a path in G around K. The mean sum for this integral is easily shown to be a rational approximation of f.

One has to make additional assumptions about K to make polynomials approximation of a function holomorphic on K possible. To see why consider the function $1/z$ defined on the set $A = \{z | r < |z| < R\}, 0 < r < R$ and let $r < K < R$. Define a path Γ by $\Gamma(t) = Ke^{it}, t \in [0, 2\pi]$ For any polynomial P we have

$$\int_\Gamma P(z)dz = 0 \qquad \text{but} \qquad \int_\Gamma \frac{1}{z}dz = 2\pi i$$

Thus a polynomial approximation is impossible. However, if we require that C$\setminus K$ is connected, then we have:

Theorem 2 (Runge's second theorem) *Let K be compact and C$\setminus K$ connected. Then any function holomorphic on K can be approximated by polynomials.*

The more general problem concerning $f \in A(K)$ was solved by S. N. Mergelyan in 1952. He proved that if K is compact with C $\setminus K$ connected then any $f \in A(K)$ can be approximated universally by polynomials on K.

Let us consider a new problem. By a domain G we mean an open connected subset of the complex plane. Let G be a bounded domain and let

$$L_\alpha^2 = \left\{ f \in \mathrm{Hol}\,(G) \Big| \int\int_G |f|^2 dxdy < \infty \right\}$$

For what G are the polynomials dense in L_α^2? By definition, a domain G will be called a Carathéodory domain iff its boundary is the boundary of some unbounded domain. For example any domain bounded by a Jordan Curve is a Carathéodory domain. We have the following theorem:

Theorem 3 (Farrel, Mapkyuecuz): *If G is a Carathéodory domain then the polynomials are dense in L_α^2.*

Consider non-Carathéodory domains like $B(0,2) \setminus B(1,1)$ If the boundary curves approach each other fast enough then the polynomials are dense in L_α^2 otherwise they are not. Thus we are not dealing with a topological property.

Let us return to uniform approximation by rational functions and define

$$R(K) = \{f \in C(K)|\ f\ \text{ has rational approximation}\}$$

Now we ask is $R(K) = C(K)$? The answer is negative. A counter example, called the *Swiss cheese*, was given by Alice Roth:

Let (z_j) be a sequence of rational complex numbers (i.e. numbers of the form $u+iv$ where u and v are rational numbers) in the closed unit disk. Let $(r_j) \subseteq \mathsf{R}_+$ have the property $\sum_{j=1}^\infty r_j < 1$ Assume that $B(z_j, r_j) \cap B(z_i, r_i) = \emptyset$ when $i \neq j$. The Swiss cheese is the set $K = \overline{B}(0,1) \setminus \cup_j B(z_j, r_j)$. By Cauchy's theorem we get

$$\int_{\partial B(0,1)} R(z)dz = \sum_j \int_{\partial B_j} R(z)dz\ ,\ (B_j = B(z_j, r_j))$$

If we have rational approximation of a continuous function f then this implies

$$\int_{\partial B(0,1)} f(z)dz = \sum_j \int_{\partial B_j} f(z)dz$$

Consider $f(z) = \frac{|z|}{z}$. The left side then equals $2\pi i$ while the right side (RS) has $|\mathrm{RS}| \leq 2\pi \sum_j r_j < 2\pi$ which is a contradiction.

The final part of the lecture was about analytic capacity and its relations to rational approximation. Let K be a compact set and assume that $\mathbb{C}\backslash K$ is connected. Let $\mathcal{A}(K) = \left\{ f \in \mathrm{Hol}\left(\widehat{\mathbb{C}}\backslash K\right) | f(\infty) = 0, \forall z : |f(z)| \leq 1 \right\}$. Let E be a subset of the complex plane. We define $\mathcal{A}(E)$ by

$$f \in \mathcal{A}(E) \Leftrightarrow f \in \mathcal{A}(K), \forall\, \text{compact } K \subseteq E$$

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For $f \in \mathcal{A}(K)$ we have the expansion $f(z) = \frac{c_1}{z} + \frac{c_2}{z^2} + \dots$ The analytic capacity is defined by

$$\gamma(E) = \sup \{|c_1(f)| \, | \, f \in \mathcal{A}(E)\}$$

It is related to rational approximation by the following theorem:

Theorem 4 (Vitushkin 1962): *Let K be a compact set. Then the following are equivalent:*

1. $R(K) = C(K)$
2. $\gamma(G \backslash K) = \gamma(G)$ *for all open G.*

The meeting with the leading staff of ILIP

After our first lecture of the week, we walked to the Palace of Youth where Genady Khorkov and Vladimir Molotsov from the leading staff of ILIP gave a speech of welcome. They were assisted by one of the Russian students, Konstantin Egorov, who did a very good job translating into English.

Mr. Khorkov (left) and Mr. Molotsov with the Euler Star

It was a warm and interesting speech. Mr. Khorkov started by telling how he and the rest of the people at ILIP were glad that the exchange project had worked out well so far, and asked us if we found our program for the week in St. Petersburg satisfactory, which we did. He thanked us for the way we had received the Russian students and explained how an exchange project could be a very important experience for the Russian and Danish students, because it meant being together and seeing the differences between

the two societies. He pointed out how important it was, that the Russian students were able to visit Denmark, something which previously would have been impossible. It was therefore a unique opportunity for them to see another society, because they belong to the new generation, which is the hope for a better future. Finally he mentioned good human relations as one of the essential parts of an exchange project and Mr. Molotsov added a few similar words by wishing us a pleasant week, and hoped that we would bring home mainly good impressions from St. Petersburg.

After the talk, the Danish students showed their gratitude by presenting the Euler Star, the logo of Eulers Venner, to the ILIP board. Once again the Russian translator showed his talent, but this time from English into Russian, when Jakob Grove, the president of Eulers Venner, explained the unique beauty of the star consisting of pyramids revealing the golden section. This explanation involved some mathematical terms, but they are after all international. After applauding for the warm welcome everybody went to lunch.

Sightseeing

Monday afternoon was the time for a sightseeing tour in St. Petersburg. We went by bus from the Palace of Youth and were guided by Andrey Solovyev, one of the Russian students. To begin with, Andrey told us about the history of the University of Electrical Engineering in St. Petersburg. It was built in 1886 and during most of its time called the Electrical Engineering Institute; in fact it was renamed only two days after the Russian students returned from Denmark. Today it consists of several departments, which together hold about 6000 students. There are two mathematical departments with about 40 teachers each.

Our guide Andrey Solovyev

Our first stop was at the Peter-Paul fortress, where we had a look at the church and the prisons. After this we continued down the Nefski Prospect, where our guide almost couldn't keep up at informing us about all the beautiful buildings and monuments. Among others, we of course also had to spend a few minutes on the great square outside the Hermitage and at the quay where the famous warship *Aurora* can still be seen.
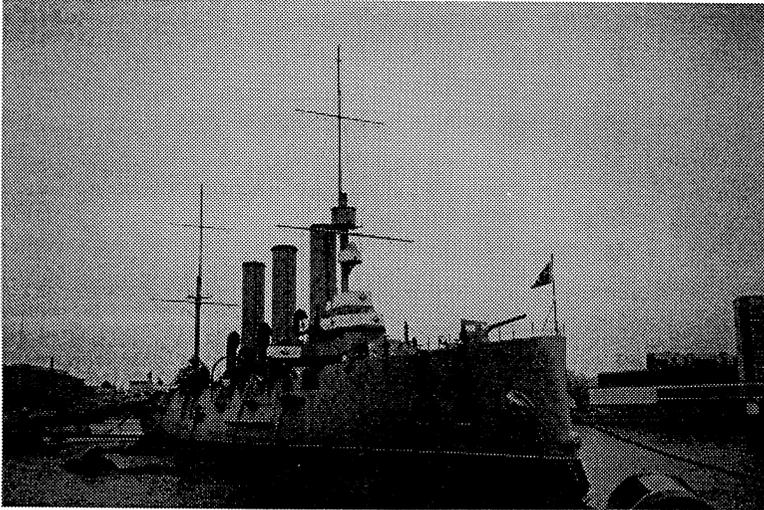
The warship *Aurora*

All in all we had a fine introduction to some of the "things to see" in the city and a lot of us returned later to have a closer look at some of the museums; e.g. the Russian Museum and the St. Isaac's Cathedral.

Tuesday 20 October 1992

8.00 — 9.00 Breakfast.
9.40 — 11.10 Lecture on state invariants on 3-manifolds.
11.10 — 12.40 Lecture on various approaches to estimates of parameters.
13.00 — 14.00 Lunch at the Palace of Youth.
14.00 — 17.00 Visit at the Zoological Museum.

State Invariants of 3-Manifolds by S. Finashin

As link-complements are rather simple (and nice) examples of 3-manifolds, the lecture by Prof. Finashin started with a discussion of invariants of knots and links. By a *knot* we will understand an embedding

$$\gamma : S^1 \rightarrow S^3$$

and by a *n-link* we will understand an embedding

$$\lambda : \coprod_{i=1}^{n} S_i^1 \rightarrow S^3$$
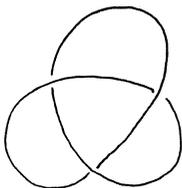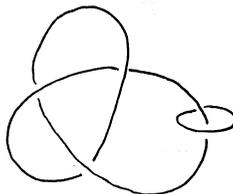
The Trefoil Knot   An arbitrary link

Links can be changed into differently *appearing* structures by means of continuous transformations. This property is called *isotopy* and induces an equivalence relation, denoted by $\sim$, on the set of all n-links. What we want is to classify links up to isotopy via their diagrams. In order to achieve this goal Prof. Finashin gave a short introduction to the Jones polynomial.

Consider a braid of, for instance, four strings:
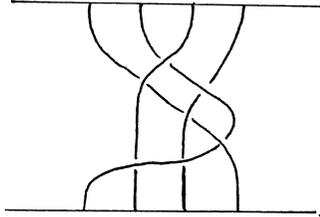
An arbitrary 4-string braid

If we take the closure of the braid; i.e. pairwise join the ends in a specific manner, we get a link. In 1928 James Alexander showed that any (oriented) link can be represented as the closure of some braid. Of course this representation is not unique:
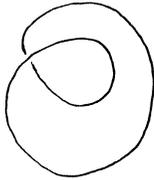
One link   Two braids

Under the operation of "concatenating braids", as illustrated below, the collection B_n of all braids on n strings constitutes a group.

Let $r_t : B_n \to N_n$ be a representation of B_n on a Von Neumann algebra N_n. The trace is a well-known invariant of maps between vector spaces. Using this fact, the composition

$$B_n \xrightarrow{r_t} N_n \xrightarrow{\text{trace}} \mathbb{C}$$

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can yield an invariant of braids, which (via Alexander's 1928 theorem) in turn gives rise to invariants of corresponding links. The Jones polynomial $V_K(t)$ was obtained in this way.

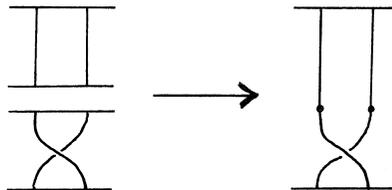The Jones polynomial is a link invariant of (ambient) isotopy and is characterized by the so-called *skein relation*. Consider a crossing point in a diagram of an arbitrary link L. We can then alter it to produce three different link diagrams as indicated below:

$$L_+ \qquad\qquad L_- \qquad\qquad L_0$$
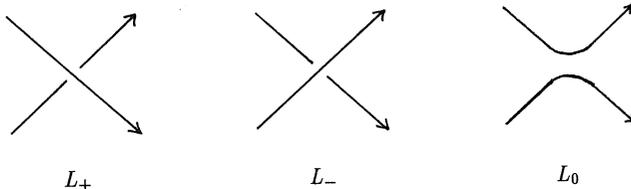
The skein relation for the Jones polynomial is then given by

$$t^{-1}V_{L_+} - tV_{L_-} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)V_{L_0} \tag{1}$$

The (ambient) isotopic invariance property of $V_K(t)$, mentioned above, expresses

$$K \sim K' \Rightarrow V_K(t) = V_{K'}(t) \tag{2}$$

and for the unknot U we have

$$V_U(t) = 1 \tag{3}$$

These three rules comprise the axiom set of $V_K(t)$. We shall use them to compute the Jones polynomial of the trivial 2–link. We shall call it L_0 in the following.

$$L_+ \qquad\qquad L_- \qquad\qquad L_0$$
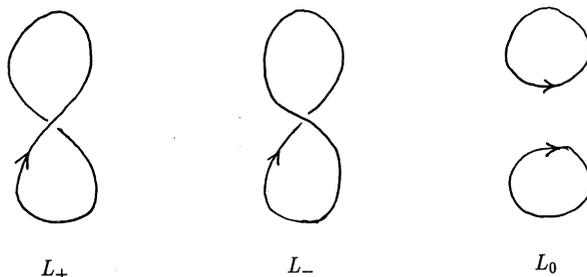
Now $L_- = L_+ = U$, and thus

$$t^{-1} - t = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)V_{L_0}$$

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this yields

$$V_{L_0}(t) = -\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)$$

After $V_K(t)$ became known, it has been generalized in two distinct ways. One way, proposed by L. Kaufmann, generalizes the Jones polynomial via state sums. The other way is the so-called HOMFLY polynomial. The name is an acronym, which is derived from the first letters of the names of 6 out of the 8 persons who (almost) simultaneously discovered it. HOMFLY is denoted by $W_K(\alpha, z)$ and satisfies the skein relation

$$\alpha W_{L_+} - \alpha^{-1} W_{L_-} = z W_{L_0}$$

It is obvious that this is a direct generalization via the coefficients in the skein relation of $V_K(t)$. HOMFLY also generalizes the quite well established Alexander polynomial, a homological invariant dating from the beginning of the 1920's and discovered by James Alexander. It became the classical trailblazer for a number of other polynomial link invariants.

The great advantage of the Jones polynomial is its superiority over previously discovered polynomial link invariants at detecting mirror symmetries. Let K^{-1} denote the mirror image of K, then the property

$$V_{K^{-1}}(t) = V_K\left(t^{-1}\right)$$

enables one to distinguish knots from their mirror image.

About three years after the discovery of the Jones polynomial a great breakthrough was achieved by Edward Witten (fall 1988), who managed to explain the physical meaning of all these invariants and simultaneously generalize W for a compact oriented 3-manifold M. In order to explain his endeavours we shall have to introduce some differential geometrical terminology.

Parallel translation affords us an identification of the tangent space at points p and q on a manifold, which are connected by a curve. A connexion ∇ induces a parallel transport of the tangent space along an arc τ. Now we are in a position to define the Witten invariant for M.

Let τ be our knot $K \subseteq M$. Using a connexion one can translate the tangent space, which yields a linear isomorphism $\tau_K : T_p M \to T_q M$. The set of all connexions on M (up to a natural equivalence relation) is denoted by C. This set forms an infinite dimensional manifold that has the structure of a Lie algebra valued field over M. Since a single knot is homeomorphic to S^1, we can calculate its so-called *holonomy*. This is a map Hol $: C \to \mathbf{R}$, which measures the directional difference of a tangent vector which has been parallel transported once around the closed path which the knot comprises. This gives an element of a representation R of $SU(N)$, that is well-defined up to conjugacy. Next we take the trace of this element in R, i.e

$$\mathcal{U}_R(K) = \mathrm{Tr}_R \left(P \exp \int_K A_i dx^i \right)$$

Let L be an n-link, with components L_i. We assign a representation R_i to each L_i and calculate the Feynman integral over the gauge field C. So we obtain the Witten invariant

$$\int D\mathcal{A} \exp\left(i\mathcal{L}\right) \prod_{i=1}^{n} \mathcal{U}_{R_i}(L_i)$$

A clear and simple definition, but it suffers some defects. For instance, what is integration? In this case one is forced to use so-called Feynmann integration; a tool which is used extensively by physicists. The problem with Feynmann integrals is that they are not properly defined from a mathematical point of view. Since they are heuristic, they cause the Witten invariant to be heuristic too. Luckily the problem of *how* to make Witten's invariant well-defined was solved by Reshetikhin and Turaev (1991). Their approach uses quantum groups to construct (generalized forms of) the invariant. Their invariants are averages over link polynomials, which are adjusted so that the resulting summation is unchanged under the so-called Kirby moves. In 1978 Kirby gave a set of moves that can be performed on framed links in S^3, which are comparable to "knotted ribbons". This caused the classification of 3-manifolds to be reduced to that of framed links up to (ambient) isotopy augmented by these moves. Reshetikhin and Turaev's work clarified the mathematical framework of the Witten invariant.

Later Turaev and Viro found variants of Witten's invariant, which could be obtained in a simpler manner. We will try and explain them briefly. The Turaev-Viro invariants are defined from a physicist's point of view by means of state sums. In statistical mechanics use is made of a certain principle. A particle p can be in any one of several states of the *state set* $S = \{1, \ldots, n\}$. The particle's energy in state $1 \le i \le n$ is denoted by E_i and is called the *state energy*. The probability for p to be in state i, i.e p_i, is given by Gibb's principle

$$p_i \sim E_i$$

The correspondence is given by

$$p_i = c\, e^{-E_i}$$

When we sum all possible state energies into the so-called *state sum*, or *partition function*, we obtain

$$Z = \sum_i e^{-E_i}$$

This principle does not only work for one single particle, but also for many. In classical mechanics there is a principle of minimal action. When subjected to change, the system behaves in such a way that the "easiest way" is followed. In statistical mechanics there is another principle, not of minimal action, but of many actions. The probability of a specific action being executed is proportional to the exponent of the energy. Let $s \in S'$, the new state set of all particles, then

$$E(s) = \sum_i E_i(s)$$

analogously for the partitioning function

$$Z = \sum_s e^{-E(s)} = \sum_s \prod_i e^{-E_i(s)}$$

Now we can say something on the definition of the Turaev-Viro invariant. Let I be a fixed set of colours and let $O \subset I^3$ be the set of (admissible) triples over $I \times I \times I$, which are ordered in a certain manner. Consider a map

$$i : I \to \mathbb{C} \backslash \{0\}$$

then the symmetric map of the so-called quantum $6j$-symbols, associated with the quantum group $sl_q(2)$ (where q is a root of unity), is denoted by

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} : I^6 \to \mathbb{C}$$

We shall also need a so-called weight ω. Representation theory of quantum groups leads naturally to these symbols, but the definition of the elements of this set of initial data should satisfy three particular relations. We shall not give them, but they are very important as they also have to ensure that the Turaev-Viro invariant does not come to depend on the choice of triangulation to be introduced next.

We can now define the state invariants for 3-manifolds. Take a triangulation of M by means of tetrahedrons, which are called *3-simplices*.
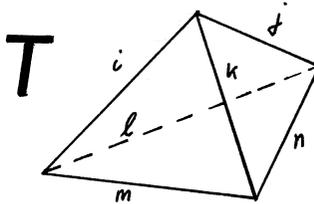
Consider the edges as particles. The total energy of the state is the sum of the energies between interacting particles. Interaction may be in one of two ways: along the connecting vertex or over a connecting face of the simplex T. These interactions relate to combinations of colourings of the 3-simplices. The latter gives

$$\prod_{F \in \{\text{faces of } T\}} \omega^{2\chi(F)}$$

where $\chi(F)$ is the Euler characteristic of F. The former yields

$$\prod_{V \in \{ \text{ of vertices of } T\}} \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}$$

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The quantum $6j$-symbols are derived from the colours of the faces intersecting at vertex V. Normalised by a factor $\omega^{2\chi(T)}$ and summed over J, the set of all admissible colourings of our simplex T, we get Turaev-Viro's invariant:

$$|M| = \sum_J \omega^{2\chi(T)} \prod_{F \in \{\text{faces of } T\}} \omega^{2\chi(F)} \prod_{V \in \{\text{vertices of } T\}} \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}$$

$|M|$ remains invariant under the so-called Matveev moves and is called the *state invariant* of M. Unfortunately there was no time for to explain something about the nice properties of this invariant, but we managed its definition anyhow.
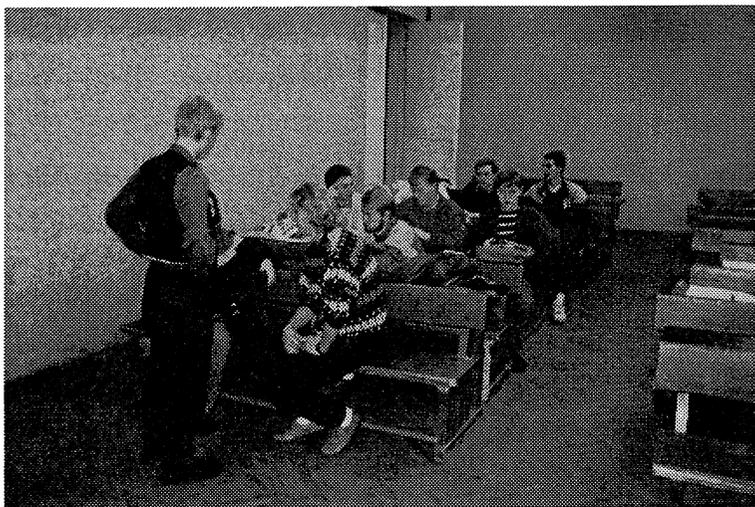
A classroom at the Electrical Engineering University

Various Statistical Approaches to Estimates of Parameters by V. Egorov

This lecture was an introduction to statistical methods. Professor Egorov started the lecture with a description of a course in statistics, that he teaches on third year. This course is divided in the following subjects:

1. Samples, Empirical estimates.
2. Task of point statistical estimation; Characteristics of estimates; Consistency; Unbiased: Efficiency.
3. Methods of finding effective estimates.
4. The information inequality.
5. Maximum likelihood estimation and least squares estimation.

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6. Linear regression.
7. Confidence intervals.

He advises his students to consult the following literature:

i. D. Bickel, K.Dokson: *Mathematical Statistics*, 1977.
ii. E. Leman: *Theory of point estimation*, 1983.

Professor Egorov then proceeded to give an introduction to the first three subjects mentioned above.

Subject 1. The first part of the lecture was about empirical estimation in general. Given a sample $x = (x_1, \ldots, x_n)$, we want to construct a corresponding distribution function F. We define the *sample distribution function*

$$F_n(z) = \frac{m(z)}{n},$$

where $m(z)$ is the number of observations less than or equal to x. We now have the following theorem:

Theorem 1 $F_n(z) \xrightarrow{p} F(z)$.

Proof: Consider

$$y_j = \begin{cases} 1, & if \ x_j < x \\ 0, & if \ x_j \geq x \end{cases} \quad , j \leq n$$

Then $F_n(z) = \frac{1}{n} \sum_{j=1}^{n} y_j \xrightarrow{p} E y_j = F(z)$ (independent of j.) $\qquad \square$

We are now in a position to construct a table of estimations of various random variables:

$$F \sim F_n$$

$$EX \sim \overline{x} = \sum_{j=1}^{n} x_j$$

$$VarX = E(X - EX)^2 \sim s^2 = \frac{1}{n} \sum_{j=1}^{n} (x_j - \overline{x})^2$$

$$EX^m \sim \frac{1}{n} \sum_{j=1}^{n} x_j^m$$

$$E(X - \overline{X})^m \sim \frac{1}{n} \sum_{j=1}^{n} (x_j - \overline{x})^2$$

$$Cov(X, Y) \sim K = \frac{1}{n} \sum_{j=1}^{n} (x_j - \overline{x})(y_j - \overline{y})$$

$$\rho \sim \eta = \frac{K}{s_x s_y}$$

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It is more difficult to estimate the density function p, corresponding to F, because F_n is discrete. We must proceed as follows. Consider a subdivision of the real line in intervals $\Delta_k = [a_k, a_{k+1}]$ and define

$$p_n(x) = \frac{F_n(a_k) - F_n(a_{k-1})}{\Delta_k} = \frac{m(a_k) - m(a_{k-1})}{n\Delta_k}$$

Now, if $\Delta_k \to 0$ then $p_n \to p$ and we are done.

Subject 2. The second part of the lecture dealt with point statistical estimation. Consider a family of distributions $g(\theta)$; $\theta \in \Xi \subseteq \mathbf{R}^k$, $k \geq 1$. We wish to find an estimator $T = T(\vec{x})$, such that the function

$$R(\theta) = E_\theta (T - g(\theta))^2$$

is minimal for any θ. In general this is not possible, but now we introduce the following definitions:

1. T is **consistent** iff $T \xrightarrow{p} g(\theta)$ for $n \to \infty$, $\forall \theta$.

2. T is **unbiased** iff $E_\theta T = g(\theta)$, $\forall \theta$.

3. T is **efficient** iff $Var_\theta T = \min Var T_1$, $\forall T_1 : E_\theta T_1 = g(\theta)$

Subject 3. In the last part of his lecture Prof. Egorov gave several examples of how to find consistent, unbiased and efficient estimates in various situations. For instance, if $g(\theta) = EX$ then $\bar{x}$ is an unbiased, and consistent estimator. Effective estimates are more difficult to construct, so Prof. Egorov introduced the notion of *sufficiency* and *completeness*, and proclaimed the following

Theorem 2 *Let T be a complete sufficient estimator for $g(\theta)$, and let T_1 be an unbiased estimator. If $E(T_1|T)$ is an effective estimator, then $T_1 = f(T)$ is also an effective estimator.*

This theorem can be used to construct efficient estimators. For example if X is uniformly distributed on $[0, \theta]$, then $x_{\max}$ is a sufficient and complete estimator, and $T = \frac{n+1}{n} x_{\max}$ is unbiased. It then follows that T is effective.

The Zoological Museum

Tuesday afternoon we went to see the Zoological Museum, which is situated just across the Neva River when standing at the Hermitage. The museum has more than 40.000 species in its impressive exhibition. We started with a guided tour, during which we were told that it is both a museum and a part of the National Academy of Science. It was founded in 1782 and has of course a large collection of species, in fact it has one of the biggest in the world. Without doubt the mammoth exhibits comprise the rarest items. There is the skeleton of a fully grown mammoth which was found in Siberia. Even more interesting is a very well preserved baby mammoth. We were told that it had fallen into a hole in the ice and died instantaneously. It froze almost immediately and

had been preserved in the ice for some 44.000 years until it was found in 1977. A lot of the animals in the museum were unfamiliar to us Danes. I think this is because the wildlife of the southern and eastern parts of the former Soviet Union is rarely exhibited in Danish zoos and zoological museums. The guide told us something very surprising about the way the museum acquired its collection of penguins: One Antarctic winter had been so cold that half a colony had died of hypothermia! That is the reason why there is a big exhibition of these birds almost in the center of the museum.
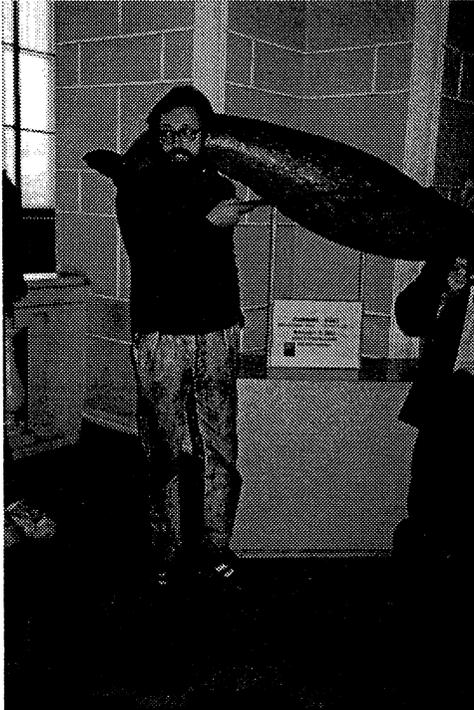
The guide at the Zoological Museum

Wednesday 21 October 1992

The Cauchy Integral by E. M. Dyn'kin

Professor Dyn'kin started off by presenting the object of his discussion; the Cauchy Operator, T. Let Γ be a closed (and nice) curve, and let $f \in L^1(\Gamma)$ be a function on an adequate subset of $\mathbb{C}$, containing Γ, then T is a linear integral operator given by

$$Tf(z) = F(z) = \int_\Gamma \frac{f(\zeta)}{(\zeta - z)} d\zeta.$$

The aim of the lecture was to discuss some properties of T.
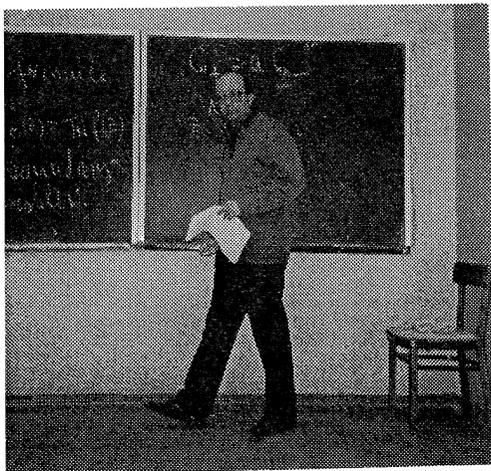
Professor Dyn'kin

1. The Cauchy formula:

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)} d\zeta$$

defines a holomorphic function outside the curve Γ. This was known by everyone, so we did not spend time by proving it.

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2. The jump formula:

$$F_i(z) - F_e(z) = 2\pi i f(z)$$

where $z \in \Gamma$ and the two functions are defined as the limits of the two holomorphic functions defined on the exterior (e) and the interior (i) of the curve.

Proof: The function $F(z) = \int_\Gamma f(\zeta)/(\zeta - z)d\zeta$ is analytic inside and outside the curve. Let G_+, G_- be the (open) interior and exterior set. We have:

$$F_i(z) = \lim_{w \to \Gamma} F(w), \quad F_e(z) = \lim_{w \to \Gamma} F(w),$$

where w converges to Γ from the inside and the outside of the curve respectively. Let $z_0 \in \Gamma$ and consider

$$Tf(z) = \int_\Gamma \frac{f(\zeta) - f(z_0)}{\zeta - z}d\zeta + \int_\Gamma \frac{f(z_0)}{\zeta - z}d\zeta.$$

Recalling the indexfunction we see that

$$f(z_0) \int_\Gamma d\zeta/(\zeta - z) = \begin{cases} 2\pi i f(x_0) & \text{if} \quad x \in G_+ \\ 0 & \text{if} \quad x \in G_- \end{cases}$$

Now look at $\int_\Gamma (f(\zeta) - f(z_0))/(\zeta - z_0)d\zeta$. Suppose our function satisfies the Hölder condition:

$$|f(\zeta) - f(z_0)| \le c|\zeta - z_0|^\alpha$$

for $0 < \alpha \le 1$, and that the curve is regular; i.e a curve satisfying the Carleman condition: $l(B(z_0, r) \cap \Gamma) \le Cr$ for every disk.
Make a partition of the curve into pieces of the following type:

$$\left\{ \zeta \in \Gamma \mid 2^k \rho \le |\zeta - z_0| \le 2^{k+1}\rho \right\}, \ k \in \mathbb{N}.$$

Now we can estimate an integral:

$$\int_\Gamma \frac{|f(\zeta) - f(z_0)|}{|\zeta - z_0|}d\zeta \le cC \left(2^N \rho \right)^\alpha,$$

where the upper bound results from summing terms from the Carleman condition up to an N large enough to cover all of the curve. So the integral is bounded and therefore convergent. Then subtraction of the following two expressions: $F_i(z_0) = 2\pi i f(z_0) + \int_\Gamma (f(\zeta) - f(z_0))/(\zeta - z_0)d\zeta$ and $F_l(z_0) = 0 + \int_\Gamma (f(\zeta) - f(z_0))/(\zeta - z_0)d\zeta$ is well-defined, and this gives the result. $\square$

3. The Riemann boundary problem.

This is a classical problem, which has some importance for physicists. We have to find two analytic functions F_+, F_- in G_+, G_- with the property

$$F_+(z) = a(z)F_-(z) + b(z),$$

where $z \in \Gamma$, $a, b \in C(\Gamma)$, $a \neq 0$. In the 1920's Torsten Carleman came up with a solution:

Suppose $b = 0$ and $a \neq 0$. Now we can find a continuous logarithm on Γ such that

$$\log\left(z^{-N}a(z)\right) \in C^\alpha(\Gamma).$$

From the jump formula, (2), we now have:

$$\log\left(z^{-N}a(z)\right) = H_+(z) - H_-(z),$$

where H_+ and H_- are analytic in their respective domains, and because of the Cauchy integral, $H_-(\infty) = 0$. Hence

$$a(z) = z^N e^{H_+} e^{-H_-}$$

(polarization). We thus have the solution:

$$F_- = e^{H_-}, \ F_+ = z^N e^{H_+}$$

For $b \neq 0$ use the first part of the construction, the jump formula and Liouville's Theorem to get the following: Let $N \geq 0$, and let $p(z)$ be a polynomial of degree at most N. Then we have

$$F_+ = e^{H_+}(P(z) + g_+(z))$$
$$F_- = z^{-N} e^{H}\left(P(z) + g_{-(z)}\right)$$

where by the jump formula: $be^{-H_+} = g_+ - g_-$.

We now move on to consider singular integral equations. Consider the eigenvalue problem:

$$(I - \lambda S)f = g$$

where f and g are supposed to be in a certain function space, S is a bounded operator (ex. the Cauchy operator), and λ is the spectralparameter. For λ small the solution is known:

$$f = \sum_{n=0}^\infty (\lambda S)^n g,$$

also called the Neumann series, after Franz Neumann.

But now a problem arises: For which curves is the Cauchy operator bounded? Of course the answer depends on the function space. This is known as the Riesz problem, which is treated in the following theorems: Let T be the Cauchy operator, then we have

Theorem 1 $T \in B(L^p(\Gamma))$, $1 < p < \infty$ *iff Γ is regular, i.e. Γ satisfies the Carleman condition.*

and

Theorem 2 $T \in B(C^\alpha(\Gamma))$,   *if Γ is regular.*

The proofs are rather long and make use of some difficult results, so we could just touch on these matters. Theorem 1 seems to be more popular and important than theorem 2. In 1977 A. Calderon showed the former for a certain class of Lipschitz curves. In 1982 Coifmann, McIntosh and Y. Meyer gave a new proof, loosening the restrictions on the curve. Finally in 1984 G. David showed a general version of theorem 1. This work was characterized by Dyn'kin as "difficult and beautiful, - but clear".

Probably not too surprising as one frequently sees that proofs, which originally are lengthy and cumbersome, are often later replaced by tricky and elegant ones. This also happened in this case: in 1987 P. Jones and S. Sommes presented a work on boundedness of Lipschitz Curves, that was "elementary, simple and clear, rendering all earlier work useless".

A sketchy outline of the main ideas in the cited works concluded the lecture. Eulers Venner's president presented Professor Dyn'kin with a gift which he received under applause of the audience.

The Electrical Engineering University

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# About the Botanical Garden and the Russian Museum

Though not included in the official program, two museums were visited Wednesday afternoon by a considerable group of Danes: the Botanical Museum just across Prof. Popova street from the University of Electrical Engineering, and the Russian Museum in the Mikhailovski Palace on the north side of the Iskusstv Square.

The idea of visiting the Botanical Museum came from one of the Danes, who has studied some biology and therefore wanted to see the Botanical Garden. His motivation for doing so was because one of its former directors, A. L. Takhtajan, during the second half of this century has contributed considerably to the present view of the evolution of plants. When we arrived, the greenhouses which we originally wanted to see were closed, but we were directed to the museum. It seemed as if this museum was mainly intended for students and researchers in botany, but we were kindly let in and the staff even provided us with a guide. Our Russian friends kindly stood for the translation. The museum contains permanent exhibitions on major botanical subdisciplines; such as plant development, evolution, plant geography and the dependence of man on plants for almost everything. Because we were short of time, we could only get some glimpses of the wealth of information in this place. The themes were displayed very instructively by a great number of alcohol conserved, dried or fossilized plants or parts of plants from all parts of the world, by plant products or by plates. Among other things we saw the strange fig species *Ficus benjamina*, which germinates in the crown of other trees and later kills its host by covering its stem with roots. When the host has perished, the roots of the fig make up a latticelike hollow "stem" of the plant. It was also thoughtprovoking to realize how deeply man depends on plants. One merely needs to mention a few species to see this: rice, wheat, corn, and cotton, and wood of course. The museum contains a great collection of wood species which in fact had to be defended during the siege of the World War II; the wood being wanted for heating.

The Botanical Museum was founded in 1823, but at that time medical herbs had been grown there for about 100 years. This part of the city is called Aptekarsky Ostrov (Apotecharys Island) for this reason.

After lunch a number of us went to the Russian Museum, one of the musts of St. Petersburg. Since we just had one afternoon we could only see a fraction of this most impressive collection of Russian art. According to the sense of location of this writer, we never left the Mikhailovsky Palace, and thus we have only been in one third of the halls of this institution. However, even this restricted opportunity to perceive some of the greatest of all art was very fascinating.

In the first rooms we saw old Russian paintings, mostly icons and art of religious content. This is art of great beauty, but also art conveying a feeling of alienation and wonder as we are dealing with an interpretation of Christianity strange to us. Maybe all kinds of religiousness are strange to modern people, but it was striking to see that most of the art of these past times was religious in contrast to more recent art, which deals with almost every aspects of life.
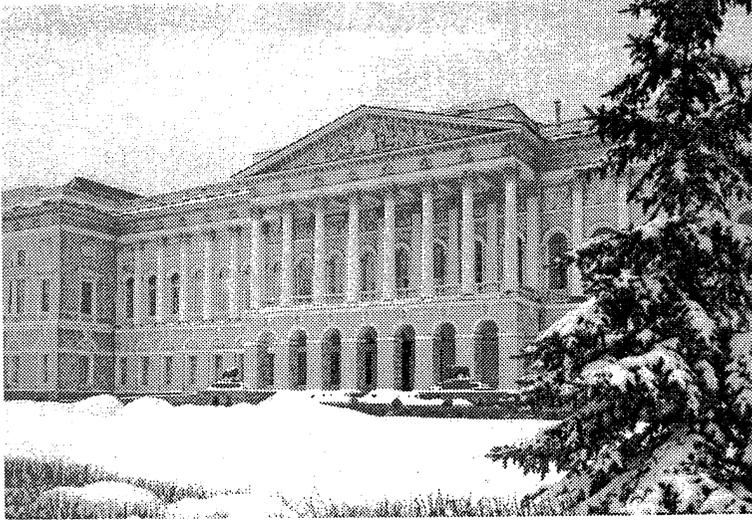
The Russian Museum

A much more direct expressive manner of painting was to be seen during the nineteenth century. As the collections are enormous only some highlights can be mentioned: The breathtaking seascape paintings by Aivazovsky, in particular *The Ninth Wave* (the ninth wave in a storm is fatal according seamen superstition), the landscapes by Shishkin, so realistic in their details that one could almost smell the bark of the pines, or feel the heat of the calm summer day in the meadows. Yet retaining their very romantic sense of the composition. Much more dramatic are the historic paintings by Briullov, by whom of course *The Last Day of Pompeii* left the deepest impression. Here one really comes to understand what an apocalypse may be like! Interestingly none of the figures on this canvas seems to be alone or forgotten. Everybody is in contact with somebody else, helping, being helped, comforting, carrying or the like. It was also instructive to see how such a picture is planned and constructed. This could be seen from the many detailed studies preceding the final work, where the artist experiments with faces, postures, etc. This was also to be seen in A. Ivanovs *The Appearance of Christ before the People* and in I. Repins *Ceremonial Meeting of the State Council*, two very different pictures, but both huge in dimensions and displaying a great number of persons, who were carefully depicted.

Many other pictures could be mentioned, but space and time restrict this author form doing so. I think everybody will agree that one single afternoon was far too little to discover anything but the most celebrated works of Russian art.

Thursday 22 October 1992

8.00 — 9.00 Breakfast.
9.30 — 11.00 Visit to LOMI: Lecture on quantum groups.
14.00 — 17.00 Visit at Pavlovsk.
18.30 — 22.00 Kirov Opera and Ballet Theatre: Mazeppa.

The Lecture on Quantum Groups by Michael Semenov-Tian-Shansky at V.A Steklov Mathematical Institute, St.Petersburg.

I will keep the level of this lecture as elementary as possible, but shall assume a basic knowledge of quantum mechanics. Currently quantum groups (QG) are a very fashionable subject. Here in St.Petersburg we pride ourselves, since in a sense their development was triggered by the work of our professors Faddeev and his colleagues. The discovery of QG is as remarkable as those of Lie groups at the beginning of the 19th century. Lie groups have proved to be ubiquitous with a wide range of applications. In particular simple Lie groups, which have been classified by Killing and Cartan at the beginning of our century. This development in mathematics represents the fulfillment of the ambitious dream of Plato, who suggested that regular polyhedra have a relation to the intrinsic structure of the universe. Throughout time people have revived and rejected this idea, but the deepness of it remained. Consider Kepler's work, which was a revival attempt which failed. However, there is a deep connection between simple Lie groups and the idea of Plato. Now we know that there are further objects such as Kac-Moody algebras and presently there is a class of objects with the same combinatorial base, but the algebraic objects are rather dissimilar. Another idea known to mathematicians long before physicists were able to interpret it, was the Schrödinger wave function, based on an equation, which mathematicians call the Stürm-Louiville Equation, known to Gauss, Legendre, Laplace, etc. What now has been realised is the rule between quantum mathematics and algebra. The mathematics required to deal with QG was at hand for a long time, such as the hypergeometric functions introduced by Gauss. What was lacking was the proper geometric interpretation.

When we speak of QG, it is an appealing expression, but what does it mean? What is in general the quantizing principle? In short it is the passing from a commutative structure to something non-commutative. We shall introduce some terminology. Let $\mathcal{M}$ be a smooth manifold, which shall represent the phase state of a system. Let $x \in \mathcal{M}$ be a point on $\mathcal{M}$, representing a state of our system. What are the observables[1] in classical mechanics? We can think of our system as a distribution of particles. The notion of an observable is a density $\varphi \in C^\infty(\mathcal{M})$. (Now φ is an observable characterizing the state of our system.) We look at a basic structure which is used in classical mechanics and called Hamilton's equation of motion (the "evolution equation"):

$$\frac{\partial \varphi}{\partial t} = \{\mathcal{H}, \varphi\}$$

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[1] Self-adjoint operators on a Hilbert space.

Then $\mathcal{H} \in C^\infty(\mathcal{M})$ is the Hamiltonian operator. The Poisson bracket $\{\}$ gives the Lie algebra structure on $C^\infty(\mathcal{M})$, having the following basic property

$$\{ab, \varphi\} = a\{b, \varphi\} + b\{a, \varphi\}$$

What happens when we pass to quantum mechanics?

$$C^\infty(\mathcal{M}) = A_0 \rightsquigarrow A_h$$

Our algebra of observables $A_0 = C^\infty(\mathcal{M})$ is replaced by a non-commutative algebra A_h. The underlying set is the same, but the composition is changed. The Poisson bracket is the guiding principle.

$$\varphi *_h \psi = \varphi\psi + \frac{h}{2}\{\varphi, \psi\} + o(h) \qquad (*)$$
$$[\varphi, \psi]_h = \varphi *_h \psi - \psi *_h \varphi = h\{\varphi, \psi\} + o(h)$$

In this approach an important fact is that we may forget about the manifold $\mathcal{M}$. All information is implicit in the observables. To have the information about x we can merely consider the subset $\mathcal{O}_x \subset C^\infty(\mathcal{M})$ of functions vanishing at x, which is an ideal since

$$\varphi \in \mathcal{O}_x \Rightarrow \psi.\varphi \in \mathcal{O}_x$$

This is an important idea, which dates from the middle of the last century, and even precedes Hilbert. The idea is that we can associate points on our manifold with ideals on $C^\infty(\mathcal{M})$. What we have used is an algebraic structure. The idea is borrowed from Algebraic Geometry and Spectral Theory.

Assume we wish to quantize *any* manifold, equipped with a Poisson bracket and pass to deformation of this enveloping algebra. It has been shown that such deformations exist, but there may be algebraic obstructions to the composition being associative. (Note that $(*)$ is only written down up to first order in h. Higher orders can be developed, but we shall return to this later.)

To get some interesting examples we shall need some more structure. Assume our manifold is a Lie group G and we have $C^\infty(G)$, which we wish to quantize. However, we also wish to preserve parts of G's structure. We first have to consider the multiplication rule of G

$$m : G \times G \to G$$

Multiplication is a map of $G \times G$ into G itself. This gives the coproduct

$$\Delta : C^\infty(G) \to C^\infty(G \times G)$$
$$\Delta\varphi(x, y) = \varphi(xy)$$

We see that if we replace G by $C^\infty(G)$ the additional structure of the coproduct appears. The properties of Δ corresponding to associativity of m can be developed. We can say

that the algebraic functor has additional structure, consisting of *Hopf Algebras*, which were discovered by Heinz Hopf.

Another essential thing is the inversion of the Lie group's element $G \to G$ given by the map $x \mapsto x^{-1}$. Dual operations on functions by the antipode S for which $(S\varphi)(x) = \varphi(x^{-1})$. To summarize, by putting everything together we have

$$(C^\infty(G), \Delta, S) = A_0$$

This data allows us to restore the group G.

Now we wish to quantize G itself. The first examples were formulated here in St.Petersburg at Steklov Institute by Drinfel'd. The deformation problem we have to deal with is listed by

1. Replace G by $(C^\infty(G), \Delta, S)$.
2. Apply the Hopf algebra structure. (Quantizing)
3. Deform it to get a QG.

Let us look at the physical reasons for deforming a Lie group. What sort of symmetry should our system have in order to be quantizable? Let us consider an example.
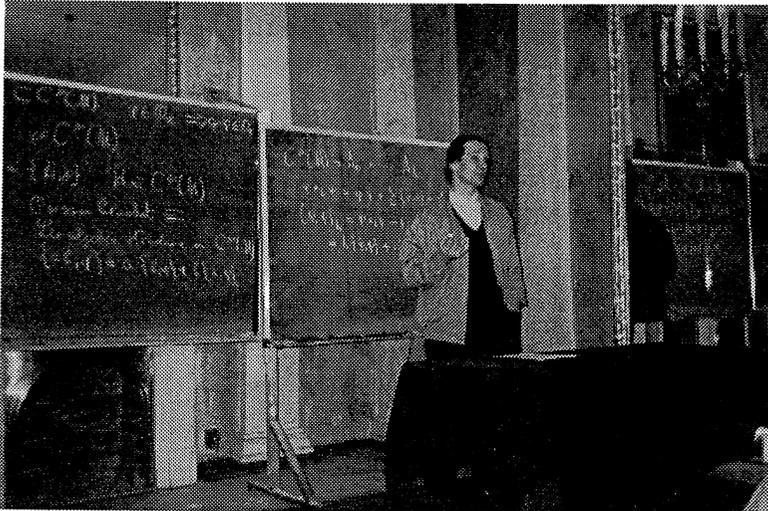
Professor Semenov-Tian-Shansky

Let $\mathcal{M} = \mathbf{R}^3$, $G = SO(3)$ and an action $G \times \mathcal{M} \to \mathcal{M}$. On the quantum mechanical level we have more than just an action. Let us assume for $x, p \in \mathbf{R}^3$ that

$$\mathcal{M} = \mathbf{R}^3 \oplus \mathbf{R}^3 = \{(x, p)\}$$

(which is an algebra of linear operators acting on $\mathbf{R}^3$ with Poisson bracket as phase space and with derivations as the *moment*)

$$\{\varphi, \psi\} = \left(\frac{\partial \varphi}{\partial p_i} \frac{\partial \psi}{\partial x^i} - \frac{\partial \psi}{\partial p_i} \frac{\partial \varphi}{\partial x^i} \right)$$

Hence our group acts in a very qualified way. G preserves the Poisson bracket and commutators upon quantization. Nothing has happened with the group itself. Merely the observables have been replaced.

Now what is different between Lie groups and their actions to validate quantizing? Ordinary symmetries which are not quantized preserve the properties of the Poisson bracket. The deformed symmetries do *not* preserve {}'s properties. This is the physical reason for considering QG in this more general setting. What deformations give the coproduct Δ? We see that we get the Poisson bracket {} as a first approximation. Hence we have a different type of (physical) symmetry. It is this idea which is at the origin of the whole theory.

Let us try to elaborate this point. There are two ways of computing a Poisson bracket. The first is

$$\{\Delta\varphi, \Delta\psi\}_{G \times G}$$

where

$$\Delta\varphi(x, y) = \varphi(xy)$$

and the other is the Poisson bracket on G itself

$$\{\varphi, \psi\}_G(xy)$$

Now we demand that these two ways must coincide, i.e

$$\{\varphi, \psi\}_G(xy) = \{\Delta\varphi, \Delta\psi\}_{G \times G}$$

This class of Poisson brackets gives rise to a Poisson-Lie group. They were discovered here in St.Petersburg and later generalised by Drinfel'd. Let us have a closer look. The group acts on itself by left translation: $y \mapsto e^{tx}y$, $x \in \mathfrak{g}$. Translation of the group does *not* preserve {}. The measure of *non*-conservation of {} by the translation is non-zero. Formal proof:

$$\hat{X}\{\varphi, \psi\} - \left\{ \hat{X}\varphi, \psi \right\} - \left\{ \varphi, \hat{X}\psi \right\} \neq 0$$

$$\hat{X}\varphi(y) = \left(\frac{d}{dt} \right)_{t=0} \varphi(e^{tx}y)$$

Hence we get the emergence of non-Hamiltonian vector fields which lead to the quantization of the Lie group. In a more general setting

$$\mathcal{M} = \mathbf{R}^3 \oplus \mathbf{R}^3 = \{(x,p)\}$$
$$G = SO(3); \qquad G \times \mathcal{M} \to \mathcal{M}$$

with Poisson brackets on $\mathcal{M}$ and G

$$\{\varphi, \psi\}(g.x) = \{\varphi(yx), \psi(yx)\}_{\mathcal{M}} + \{\varphi(yx), \psi(yx)\}_G$$

where the first term is a constant and the second term, due to the Poisson brackets on the Lie group itself, tells us that the Poisson brackets are non-conservative. Now the basic idea is to find non-trivial transformations which preserve $\{\}$. To proceed we must first tackle the problems at the semi-classical level: classify Poisson brackets and then quantize them.

Simple example. Let $G = SL(2)$ and $C^\infty(G) \subset P(G) = \mathbf{C}[a, b, c, d]$, where $P(G)$ are all polynomials on G. An arbitrary element $T \in G$

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (t_{ij})$$

satisfies $\det T = (ad - bc) = 1$. Now we will try to quantize this matrix algebra. First we define a coproduct

$$\Delta : P(G) \to P(G) \otimes P(G)$$
$$\Delta t_{ij} = \sum_k t_{ik} \otimes t_{kj}$$

We do so by dualizing the matrix multiplication (i.e simple matrix multiplication in different guise). Before quantization on our algebra occurs we have

$$A_0 = P(G) = \mathbf{C}[a, b, c, d]$$
$$t_{ij}t_{kl} = t_{kl}t_{ij}$$
$$T_1 \otimes T_2 = T_2 \otimes T_1$$

The commutativity of matrix coefficients is denoted by the Kronecker product of 2 matrices. Hence A_0 is commutative with point-wise multiplication of matrix coefficients. We can write down the full quantum answer, making a QG by the following remarkable property:

$$T_2 \otimes T_1 = R_h(T_1 \otimes T_2)R_h^{-1} \qquad (**)$$

The Kronecker product, written in different order, is not any longer commutative, but "conjugative". A solution to (**) is $R \in \text{End}(\mathbf{R}^2 \otimes \mathbf{R}^2)$, where R is a 4×4 triangular matrix, taking the following form

$$R = \begin{pmatrix} e^h & & & 0 \\ 0 & 1 & & \\ 0 & e^h - e^{-h} & 1 & \\ 0 & 0 & 0 & e^h \end{pmatrix}$$

In our algebra A_h the comultiplication satisfies $(**)$. We can get this confirmed by subjecting the matrix entries to a check:

$$\Delta t_{ij} = \sum t'_{ik} \otimes t''_{kj}$$
$$T = T'T''$$
$$T'_2 T'_1 = R T'_1 T'_2 R^{-1}$$
$$T'_1 T''_2 = T''_2 T'_1$$
$$T''_2 T''_1 = R T''_1 T''_2 R^{-1}$$

As is well-known, $SL(2)$ acts on the linear transformations of the plane. Analogously our QG acts on linear transformations of the quantum plane $P_h(x_1 x_2)$

$$x_1 x_2 = e^h x_2 x_1$$
$$x_i x_j = e^h x_j x_i \qquad i < j$$

(Herman Weyl algebra) We can define an action of the group as follows:

$$x'_i = \delta(x_i) = \sum_k t_{ij} x_k$$

(Ordinary matrix multiplication formulas. Multiply x by matrix t from the left). Now x'_i satisfies the same commutation relation. This is easy to prove via the constraint on the quantum determinant:

$$\det{}_q T = ad - e^h bc = 1$$

The Planck constant e^h is expandable in powerseries of h. The first terms give rise to the Poisson bracket. Unfortunately I have no time left to discuss the deep properties of R which encodes all information of our system, but in a sense **a good quantization is almost unique**. If, for instance, t' commutes with t'' then this will give rise to another matrix R'. Any arbitrary R' yields a trivial QG. So R is quite special.

As said parts of the quantum mathematics were known to Gauss, in particular the calculus behind QG. Consider $SO(3)$, its actions on $\mathbf{R}^3$ and

$$\Delta = \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

which is known as the Casimir operator with infinitesimal translation. It yields the Legendre Equation

$$L\varphi = \frac{d}{dz}\left(1 - z^2\right)\frac{d}{dz}$$

Its solutions are the Legendre polynomials, which have to do with the representation of the corresponding Lie group, in particular the irreducible ones. Now we can ask similar

questions for the QG. What are the similarly associated equations? The answer is quite exciting indeed: the equations are not differential but *differences* and are given by

$$D\varphi(z) = \frac{\varphi(e^h z) - \varphi(e^{-h} z)}{e^h z - e^{-h} z}$$
$$\mathcal{L}\varphi = Dz(1 - z)D\varphi$$

Where Gauss' q-difference operators are $q = e^h$. Gauss' choice of the letter q almost shows that he anticipated these developments! One can show that 0-invariant transformations are replaced by q-differences in the quantum cases. So we need only look at old textbooks to expand our new theory.

In conclusion a few words on the application of quantum groups. Their very nature seems to suggest that we may have to introduce a notion reminiscent of "discrete space-time", or perhaps discretized versions of it. Finally there is the connection to Knot Theory. Quantum groups, emerging via solutions to the so-called Yang-Baxter Equation, which is a crucial equation in completely solvable models occurring in quantum mechanics, yield invariants of knots and links in particular but also for the more general 3-manifolds.

The Questions after the Lecture and the guided Tour round the Institute

After his lecture Mr. Semenov-Tian-Shansky gave us the opportunity to ask some questions. The first of the questions was whether he could tell us about the structure of the institute? It is small, exceptional and created in the late 1920's. It was then called the physics-mathematical institute. The physics part moved to Moscow in the 30's and has become an enormous body engaging thousands of scientists. However neither the mathematical limb of the Steklov Institute in Moscow nor St. Petersburg is very large. Here in St.Petersburg 10–15 mathematicians are employed. They work on number theory, geometry, function theory, numerical methods and physical mathematics. I work in mathematical problems in physics. Steklov is small for several reasons. Currently half of our capacity has moved to the West. This cannot be called favourable for the students. This is a research institute, but all students are welcome to participate at the weekly seminars. Students which are accepted at Steklov are trained here and eventually do their own research. Ofcourse we need some "critical mass" to tackle the problems that brings along. We are lucky to have some students at our institute to ensure our future. The work at the institute is very liberal. It is expected that we *do* a mathematical job and are not just sitting around doing what we want. This allows for an effective style of work. However we do not have place enough to accommodate all. Some work at home, but I would call the style at the institute, on the whole, liberal and effective.

Wherefrom do the students originate? Students are in a difficult situation. Most universities have moved out of town. That caused a considerable split in teaching and research work. We are working at that problem. We do not have all the people we need for teaching. Hence our policy is that we accept the brightest students. They should either prove themselves bright or pass a series of exams. Students attending

seminars must write reports. This offers us a possibility to evaluate them. Every year 2–3 students, as postgraduates, are invited to the institute.
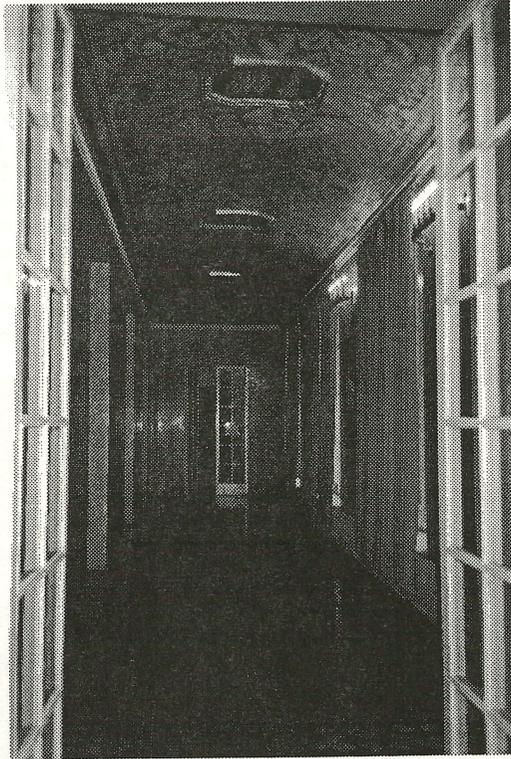

The hall of the conference room

The final question, before we were shown around the institute, was to explain a little more about the "discreteness of space-time" he had hinted at in his lecture. He said that, in the realms of QG, so far all fundamental invariant operators have been shown to be *difference* equations. However, there is at present, no natural way to introduce discrete space time. Quantum groups only give very strong hints.

The institute is a former palace. Unfortunately we could not visit the library, but Mr. Semenov-Tian-Shansky showed us one of the other classrooms and the institute's conference room. At the end of our trip we were given a set of LOMI preprints to bring back to the library of Matematisk Institut.

Pavlovsk

In the afternoon we went outside the city by bus to a wonderful place called Pavlovsk, the latest palace of the tsars in the surroundings of St. Petersburg. The story of Pavlovsk starts in the year of 1777 when the estate changed hands, as a gift, from Catherine the Great to her son and daughter in law Paul and Maria Feodorovna. It consisted of 977 acres with woods, ploughed lands and two small villages with peasants. The construction of Pavlovsk itself began in 1781 by the Scottish architect Charles Cameron. It is composed of a cubeshaped central three story building, crowned with a low dome, encircled by 64 slender columns, flanked by two single-story colonnades of curved open-winged galleries that linked it to one-and-a-half-story servants residences. Pavlovsk was the lifework of Marie Feodorovna, who was a talented woman. Among other things she spoke 6 languages and had a good sense of mathematics. Pavlovsk was her creation to which she applied all her considerable artistic ability and tireless energy. No palace bears the stamp of its owner more. Marie Feodorovna gave the palace 40 years of her life, and there she left all that was best and most characteristic of herself.
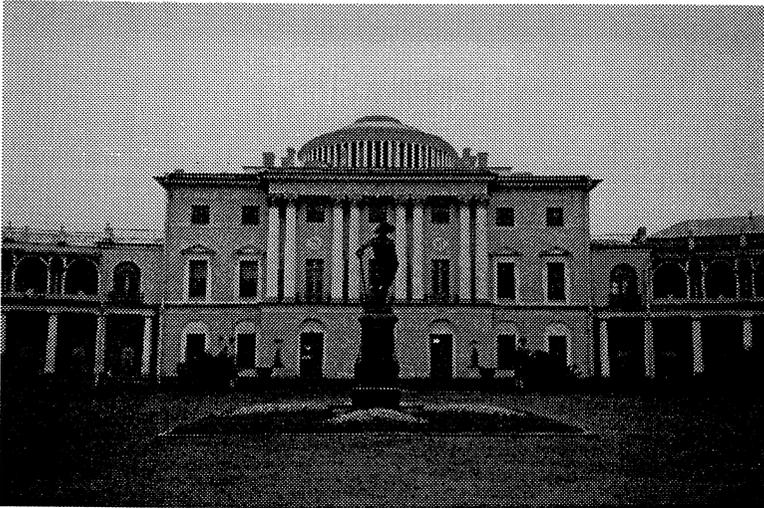
Pavlovsk

Before we were guided through the marvellous rooms, we had to put on some special shoes to prevent the floors from being worn down too fast. There was much French furniture in the rooms and the guide explained us that French simply was "in" at that time. At first the guide thought we were Italians and therefore was a bit surprised, that we were so blond, but there was a simple explanation; we were Danes! During the second world war the Germans used Pavlovsk as a headquarter. On January 24th

1944 Pavlovsk was liberated by Russians but before the Germans were defeated they succeeded in burning down the palace. Even though the palace was burned down almost to the ground and the park was filled up with bunkers, barracks and minefields, the Russians still had faith in its restoration. By now people have been working 45 years on the palace, and today one can walk through the halls, which have been completely restored and make you feel that you are back in the 18th century. It is no wonder that Pavlovsk has been called the finest restoration in the world.

We concluded our visit at Pavlovsk with a little stroll through the magnificent park. However, by this time it was getting dark and we had to return to the city for the evening's cultural event. On our way we stopped at a Russian orthodox church where we saw an authentic Russian sermon.

The Opera

On Thursday evening we saw Tjajkovskij's opera "Mazeppa" with texts by Pushkin in the Kirov Theatre. The Kirov Theatre is the most famous theatre of St. Petersburg, and here all the famous Russian operas are given their first performances. The opera is a dramatic love-story about Mazeppa, a Cossack headman.

1. act. The opera begins with Mazeppa visiting his friend landowner Kotshubei. Kotshubei's daughter Maria and the ageing Mazeppa fall in love with each other. Andrei, a friend of Maria's youth, listens to Maria talking to herself. He loves her deeply, but Maria cannot return his love because her heart belongs to Mazeppa.

When Mazeppa and Kotshubei step out of the house people start singing and dancing in honour of the guest. Mazeppa proposes to Maria, but her father refuses to give her away. He reminds Mazeppa of his age. The lovers will not accept this, and they run away together.

The second scene takes place in a room in Kotshubei's home where Maria's mother is longing for her daughter. But the landowner initiates his closest friends in his plan to take revenge on Mazeppa. Kotshubei says that Mazeppa intends to join forces with the Swedish king and secede Ukraine from the great Russian empire. In order to prevent Mazeppa's treason the young Andrei is sent to warn the tsar. They hope this will mean the execution of Mazeppa.

2. act. In the third scene we see Kotshubei kept as prisoner in a cellar by Mazeppa. This happened because Tsar Peter I did not believe Kotshubei's story. Mazeppa orders Kotshubei to be tortured and executed the next morning. However, the landowner is not afraid to die. He is just sorry he cannot take revenge on Mazeppa.

We see Orlik (Mazeppa's best friend) interrogate Kotshubei. He wants to know where Kotshubei hides his treasures. Maria's father says that he has only three treasures: his daughter, his honour and the holy revenge. The first two were stolen by Mazeppa. The last one, which he could not trust to give to anyone, he would take with him into his grave.

The next scene takes place in Mazeppa's palace. Mazeppa has a bad conscience. How can he tell Maria that her father will be executed? When Maria enters, Mazeppa confides his ambitious plan to support the Swedish and Polish king to her. She is dazzled by the thought of seeing Mazeppa as tsar of Ukraine and she promises that she is ready to risk her life and to abandon her father. However, later on she meets her mother, who tells her about Kotshubei's destiny, and she runs away to find Mazeppa.

In the fifth scene people are gathered near a wall, and Kotshubei is to be executed.

3. act. First we hear the symphony orchestra playing "The battle of Poltawa", and then the curtain raises for the last scene, which takes place at Kotshubei's estate.

Everything shows traces of war. Mazeppa had to flee after the revealing of his treacherous plans. We see Andrei looking for Mazeppa, and when Mazeppa turns up he wounds Andrei fatally in a duel.

Then Maria enters; she is absent minded and does not recognize Mazeppa. She stays with Andrei who dies in her arms. In her insanity Maria sings a touching lullaby.

Friday 23 October 1992

The Visit at the Hermitage

Our guide introduced the Hermitage by telling, that it is one of the greatest art museums in the world, comparable to Louvre in Paris, the British Museum in London and the Metropolitan in New York. It consists of 4 buildings: the Small Hermitage, the Old Hermitage, the New Hermitage and the Winter Palace. The number of halls varies, depending on the source of information, but is some where between 400 and 1000.
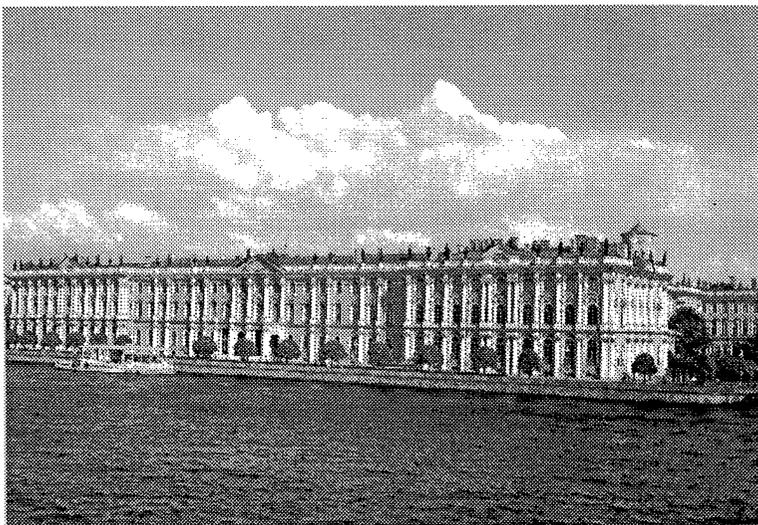
The Winter Palace, a part of the Hermitage

First we were guided through several very impressive halls, heavily decorated with gold and marble, and containing porcelain and silver. The collection of silver is bigger than the one in Paris. We were lead through a long hallway filled with portraits of participants in the war against Napoleon, and came to the halls of Impressionists and Post Impressionists. The guide told us a little about the paintings, and gave us 15 minutes to look at the exhibits. It is the world's second largest collection after the Louvre. Considering the displays incorporated works by artists such as Renoir, Cezanne,
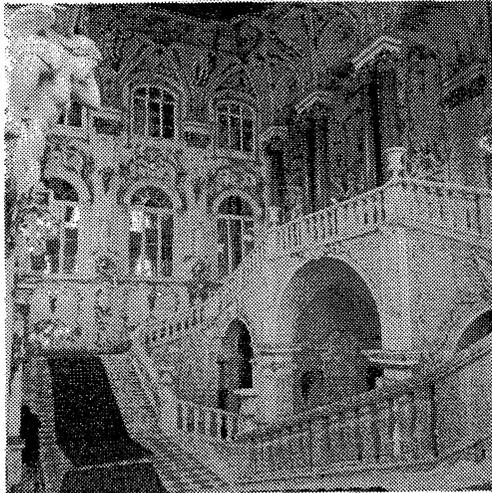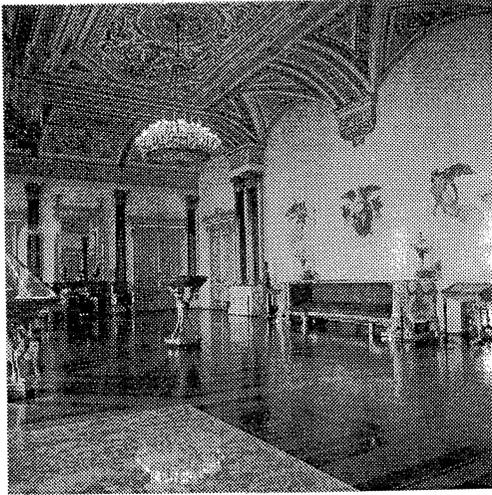
Halls of the Hermitage

Picasso, Matisse, Gauguin, van Gogh and Monet, we were not given much time for artisans of such high class.

Then the guide took us through rooms with names like The White Room, The Golden Room and The Conference Room, through a hallway with tapestries, all very beautiful. That carried us to The Small Hermitage, the interior of which consists of columns, chandeliers, and the omnipresent gold. In this building some of the most magnificent objects of art were on display: 2 of the 12 paintings by Leonardo da Vinci known to exist in the world, paintings of Madonna, 2 works by Rafael, one Madonna, his first, painted when he was only 19 years old, and a painting of The Holy Family, and finally a sculpture made by Michelangelo.

Then we went to the hall of Rembrandt and his school. We were given 8 minutes to admire the works. The Hermitage possesses 1300 Dutch paintings, including 42 works by Rubens, and thereby has the second largest collection of Dutch art in the world.

The Farewell Party at the Palace of Youth Hotel

We had a fine dinner. Knowing each other well, the party was quite lively. At 21.00 we went to the so-called Grill-bar, where a band was playing and people were dancing. Suddenly a brand new casino was discovered right next door. Having learned about probability theory, some of us saw ourselves as participants at the roulette table within no time. One must conclude that in this particular game brains don't count. Others continued the dance till late.
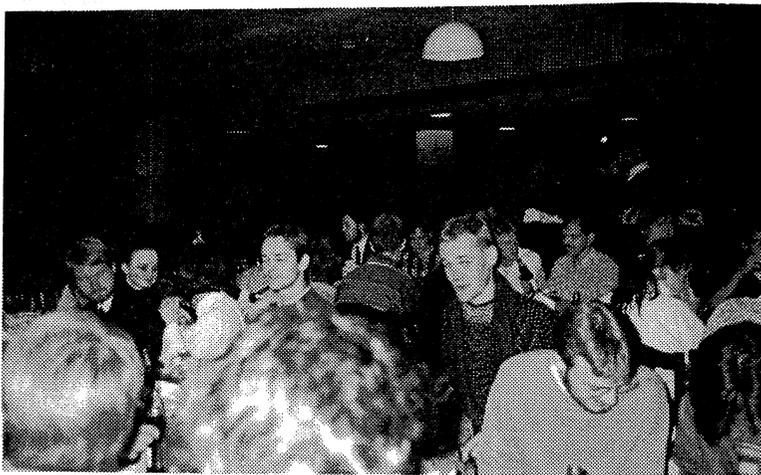
The party at the Palace of Youth

Saturday 24 October 1992

All saturday was free for people to do what they wanted. Contrary to the beginning of the week, when we did not know the city that well, people were now able to manage on their own. We spent the day in smaller groups, Danes and Russian "guides". Many went shopping, some went to museums (of which St. Petersburg seems to have infinitely many!), others rested at the hostel. In the evening one group went to a restaurant, and a disco afterwards, others revisited the new-found Casino, and others again enjoyed a last evening in the very hospitable Russian homes.

Having to leave in the morning, everybody was home early.

The Palace of Youth

Sunday 25 October 1992

Sunday was "say good-bye" day. It was time to leave our new friends in St. Petersburg, and return to Denmark, and Aarhus Universitet. Some of us wanted to stay, to see more of this very big beautiful city. Anyhow, at nine o'clock everybody had come to the youth hostel, both the Danes (luckily!) and some of the Russians, who wanted to accompany us to the airport.

The departure department of the airport is fairly new and modern. We were told, that when a new section was added onto the airport, the old part was to be used for international traffic. The new department would be reserved for domestic travel. Anyhow, we passed the custom, passport and security check within an hour, and had only a few problems at the customs, though some of us were heavily loaded with souvenirs!
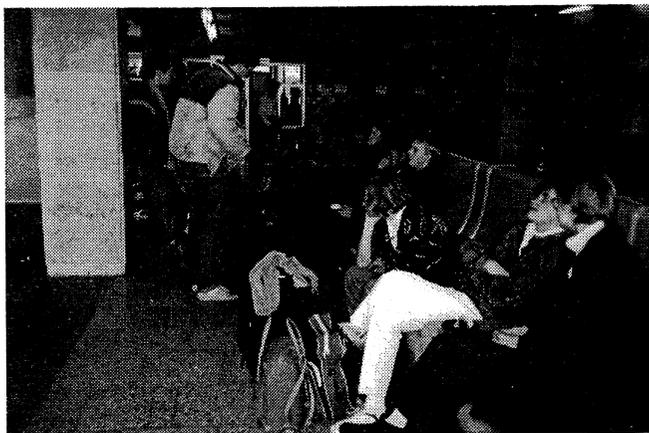
Tired Danes at the airport of St. Petersburg

To make the flight home a bit more exiting, Aeroflot had found an even smaller plane for us this time! Our ticket promised us tree motors, but this model had only two. Well, it had two wings, and we did come home safely. In fact the cabin was less noisy than our first Aeroflot plane. Many a fine morbid joke was invented on this good occasion. Some of us enjoyed them more than others. The view was good on our way home, since clouds were few. At Kastrup airport everything went smooth, we were through the control and already on the bus within fifteen minutes. We surely were home again; where things could be expected to work well and cost a lot of money. However, we had to wait one and a half hour at Copenhagen station before our train left. Being pretty tired, the traintrip was quiet and pleasant. At 6.15 pm we were in Århus again. One week of exiting experiences had ended.

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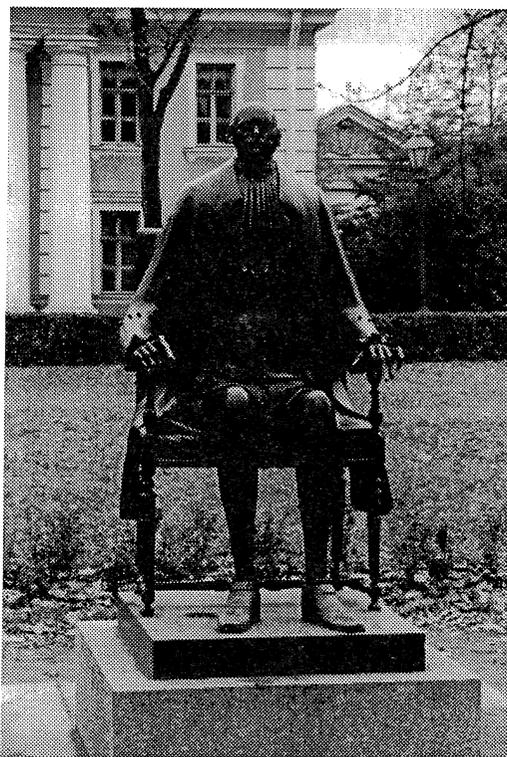
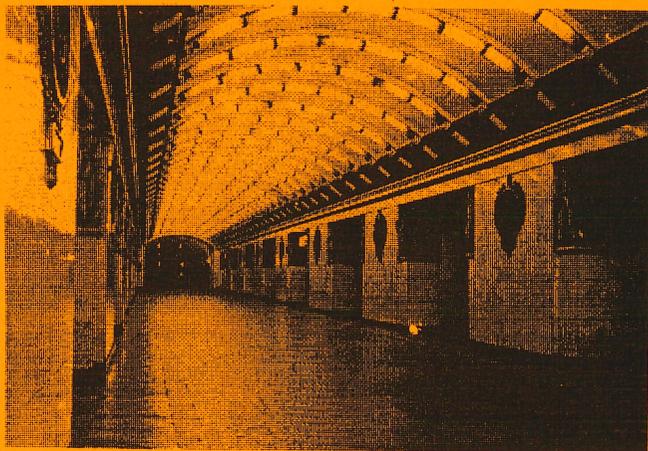Statue of Peter the Great, the founder of St. Petersburg

List of Participants

Russian Participants

1. Yekaterina Nikolayevna Mustafyna
2. Stanislav Alexandrovich Krainikov
3. Konstantin Georgyevich Egorov
4. Roman Nikolayevich Gorodnitchenko
5. Alexey Valeryevich Pertchukov
6. Yaroslav Yevgenyevich Suchkov
7. Pavel Alexandrovich Zharov
8. Vladimir Juzephovich Parkman
9. Elena Efimovna Roubintshik
10. Mikhail Victorovich Kolesnik
11. Alexey Victorovich Radionenko
12. Natalya Sergeyevna Tomashkova
13. Alexey Alexandrovich Senichev
14. Andrey Olegovich Solovyev
15. Denis Alexandrovich Prourzin
16. Olga Nikolayevna Yermilova
17. Maria Alexandrovna Victorova
18. Elena Alexandrovna Tikhonova
19. Anna Vladimirovna Podolskaya
20. Serguey Maratovich Guryev
21. Alexandr Igorevich Shramkov
22. Olga Borisovna Michailova
23. Yevgeny Borisovich Shapoval
24. Alexandra Vladimirovna Matveyeva
25. Marina Alexandrovna Robachevskaya
26. Konstantin Mikhailovich Dyakonov
27. Tatyana Vladimirovna Shulzhenko
28. Yevgenia Albertovna Safarova
29. Timophey Sumyn

Danish Participants

1. Pieter van de Griend
2. Morten Brun
3. Nils Byrial Andersen
4. Thomas Østergaard
5. Anette 𝔊𝔠𝔥𝔲𝔩𝔷 Petersen
6. Thorkild Jensen
7. Terese M. O. Nielsen
8. Christian Schlichtkrull
9. Bjarne Aagaard
10. Christian Holst Hansen
11. Henrik Just
12. Morten Storm
13. Søren Fournais
14. Niels Eriksen
15. Lars P. G. Westergaard
16. Iver M. Ottesen
17. Anne Kirstine Simonsen
18. Elise Christiansen
19. Diana Nielsen
20. Niels Henriksen
21. Eva Kathrine Petersen
22. Johan P. Hansen
23. Thomas Kjellberg Christensen
24. Gorm Salomonsen
25. Jakob Grove
26. Nina Bargisen
27. Jacob Schach Møller

Осторожно! Дверы Закрываются!

St. Petersburg Metro